

# How to hunt wild constants

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February 4, 2022

## Abstract

There are now several comprehensive web applications, stand-alone computer programs and computer algebra functions that, given a floating point number such as 6.518670730718491, can return concise nonfloat constants such as  $3 \arctan 2 + \ln 9 + 1$  that closely approximate the float. This is analogous to unscrambling an egg. Such software includes AskConstants, Inverse Symbolic Calculator, the Maple identify function, MESearch, OEIS, Ordner, RIES, and WolframAlpha. Usefully often such a result is the exact limit as the float is computed with increasing precision. Therefore these program results are candidates for proving an exact result that you could not derive or conjecture without the program. Moreover, candidates that are *not* the exact limit can be provable bounds, or convey qualitative insight, or suggest series that they truncate, or provide sufficiently close efficient approximations for subsequent computation. This article describes some of these programs, how they work, and how best to use each of them. Almost everyone who uses or should use mathematical software can benefit from acquaintance with several such programs, because these programs differ in the sets of constants that they can return.

## 1 Introduction

“... *you are in a state of constant learning.*”

– Bruce Lee

This article is about numerical *mathematical* constants that can be computed approximately, rather than about dimensionless or dimensioned physical constants. This article is a more detailed version of the conference presentation [21].

For real-world problems, we often cannot directly derive exact closed-form results even with the help of computer algebra, but we more often can compute approximate floating-point results – hereinafter called **floats**. For some such cases there is an exact closed-form expression that the float approximates, and that expression is simple enough so that we would like to know it, but we do not know how to derive it or to guess it as a prerequisite to a proof. If we had one or a few plausible concise nonfloat candidates

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that agree sufficiently closely with the float, then we could concentrate our efforts on attempting to prove that one of those candidates is the exact result.

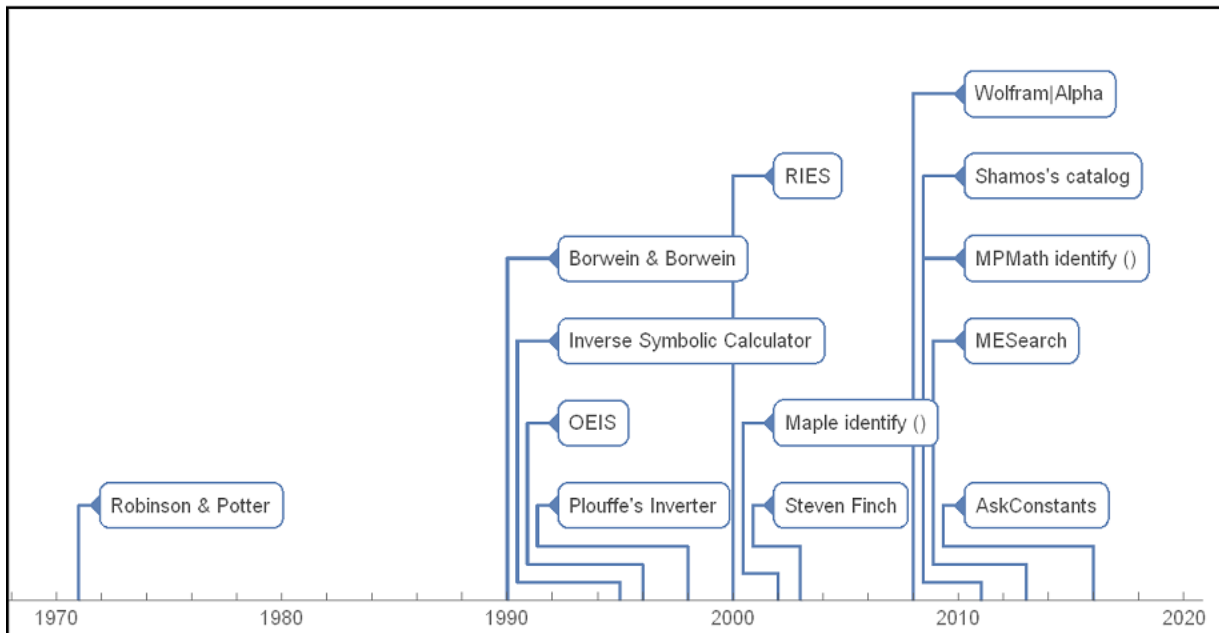
This article describes text and software tools that provide such candidates. Table 1 lists several such tools in order of presentation, grouped by type. Figure 1 plots the initial publication dates of these tools on a time line. All of the web-based and downloadable tools are free, but some of the downloadable tools require Java, a C compiler, Maple or *Mathematica*.<sup>1</sup>

Table 1: Tables in books, web apps, standalone apps, CAS functions, and CAS apps

Type	Name	Sec.	Needs	<a href="http://www.">http://www.</a> or <a href="https://www.">https://www.</a>
<b>Book table</b>	Robinson & Potter	2.1	Library or buy	<a href="http://escholarship.org/uc/item/2t95c0bp">escholarship.org/uc/item/2t95c0bp</a>
	Borwein & Borwein	2.2		<a href="http://springer.com/gp/book/9781461585121">springer.com/gp/book/9781461585121</a>
	Steven Finch	2.2		<a href="http://sites.oxy.edu/lengyel/originals/0521818052ws.pdf">sites.oxy.edu/lengyel/originals/0521818052ws.pdf</a>
<b>.pdf table</b>	Shamo's Catalog	3.1		[20]
<b>Web search</b>	Google, etc.	4	Smart-phone	<a href="http://Google.com">Google.com</a> , etc.
<b>Web apps</b>	OEIS	5.1	or computer	<a href="http://oeis.org">oeis.org</a>
	Ordner	5.2		<a href="http://fungrim.org/ordner">fungrim.org/ordner</a>
	WolframAlpha	5.3		<a href="http://wolframalpha.com">wolframalpha.com</a>
	ISC	5.4		<a href="http://wayback.cecm.sfu.ca/projects/ISC/ISCmain.html">wayback.cecm.sfu.ca/projects/ISC/ISCmain.html</a>
	ISC+	5.5		<a href="http://isc.carma.newcastle.edu.au/">isc.carma.newcastle.edu.au/</a>
<b>Stand alone</b>	MESearch	6.1	Java	<a href="http://plouffe.fr/MESearch/">http://plouffe.fr/MESearch/</a>
	RIES	6.2	C compiler	<a href="http://mrob.com/ries">mrob.com/ries</a>
<b>In CAS</b>	identify	7.1	Maple	<a href="http://maplesoft.com/support/help/Maple/view.aspx">maplesoft.com/support/help/Maple/view.aspx</a>
	identify, findpoly	7.2	SymPy	<a href="http://docs.sympy.org/0.7.1/modules/mpmath/">docs.sympy.org/0.7.1/modules/mpmath/</a>
<b>CAS apps</b>	AskConstants	8.2	<i>Mathematica</i>	<a href="http://AskConstants.org">AskConstants.org</a>
	Plouffe's Inverter	8.3	Maple	forthcoming: <a href="http://archive.org">archive.org</a>
<b>PSLQ</b>	Custom IR model	9	Fortran C++ or CAS	[1, 2]

<sup>1</sup>*Mathematica* is free and pre-installed on some Raspberry Pi computers costing only a few dollars.

Figure 1: Tables, web apps, download apps and functions to propose exact constants



Float inputs for these tools often arise from computations such as numerical integration, infinite series, iterative equation solving, approximate optimization, etc. The purposes of this article are to explain how best to use these tools, to explain how they work, and to explain how to decide if proposed candidates are promising or else probably impostors.

Section 8 discusses input magnitude limitations and Section 9 discusses some common causes of impostor results, with conclusions in Section 10.

## 2 Text tables

### 2.1 *Mathematical Constants* table by Robinson & Potter

Here is an excerpt from a hand-typed table of about 3000 constants by Robinson and Potter [17]:

...	...
4 .22755 35333 76265 40809	$-\psi(1/4) = \gamma + 3 \ln 2 + \pi/2$
0 .22755 09577 68849 99385	$4/(\pi^2 e^\gamma)$
0 .22756 34054 87472 14332	root of $7xe^x = 2$
...	...

Notice that:

- Inverse to more common tables of constants, the inputs in the left column are floats and the expressions in the right column are corresponding exact constants.
- The digamma function of  $1/4$  was negated to make the float positive. Most such tables do this **sign aliasing**, because it is easy to discard the sign of the float, then negate the corresponding nonfloat result. This doubles the potential coverage without increasing the table size.
- The inputs are sorted by the *fractional parts* of the absolute values of the input floats rather than by those entire input floats. For a given float  $\tilde{x}$ , you do a manual search for the fractional part of  $|\tilde{x}|$ , then decide whether or not the discrepancy with the fractional part of either bracketing entry or of entries slightly further away is small enough to justify further consideration. This **fractional-part aliasing** makes the table applicable to many more examples and is easily inverted mentally to construct the true candidate. For example, from the table float 4 .22755 35333 76265 40809, we can also easily guess that a numerical result 5.22755 35332 might approach  $1 - \psi(1/4)$  as the precision increases.
- The last entry in the above table excerpt is an implicit result. Since the popularization of the Lambert  $W$  function [5], this result can now be expressed explicitly as  $W_0(2/7)$ , but the table contains approximate solutions to other equations that cannot yet be expressed explicitly, together with approximate definite integrals, infinite series, and infinite products having no known closed form.

To view a photocopy of the entire table, visit

<https://escholarship.org/uc/item/2t95c0bp>

**Definition. Published constants** are publicly accessible closed-form and/or approximate float constants – either printed or on the web.

**Definition. Named constants** are constants having a widely-accepted name, such as Catalan’s constant or the *twin prime* constant. Named constants include those with an OEIS name as described in subsection 5.1.

**Definition. Tabulated constants** are formed by systematically applying sets of functions to systematic sets of arguments, Printed function tables for approximate computation usually have equally-spaced arguments that are terminating decimal fractions such as 1.001, 1.002, 1.003, etc. In contrast, to return nonfloat results such as  $\arctan(2/3)$ , the curated and computer-generated tables described in this article often instead use the set of all reduced fraction arguments whose numerator and denominator magnitudes do not exceed some given integer – perhaps also multiplied by common irrational constants such as  $\pi$ ,  $\sqrt{2}$ , etc.

*Remark.* The table by Robinson and Potter contains all three of the these types.

**Definition. Wild constants** are computed float constants for which **you** do not know an exact closed form for the limit as the precision of the float approaches infinity.

## 2.2 Other printed text tables

- Borwein and Borwein authored a similar printed table of about 100,000 constants [3]. The table contains mostly computer-generated tabulated constants and some dimensionless physical constants.
- The award-winning book by Steven Finch [9] has not only a table of about 10,000 well chosen non-tabulated constants, but it and Volume II [10] contain short essays about those constants, with references to the literature about them.

## 3 Downloadable .pdf and web-searchable tables

### 3.1 Shamos's Catalog of the Real Numbers

Michael Shamos posted a .pdf file of a table of about 10,000 constants at <http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.366.9997&rep=rep1&type=pdf>. Most of these constants are isolated pairs of a float value and a closed form defined by a definite integral, infinite series, or infinite product. Many floats list several corresponding formulas, such as various integrals, series and products that have the same value, which is helpful by suggesting other perhaps more efficient ways to compute more digits.

A web search for “**Mathematical constants**” can locate other tables that can be downloaded or searched directly on the web.

## 4 Web browsers

General-purpose web browser search engines are also helpful for directly finding candidate nonfloats for particular float inputs.

Here are some **input tips** for using the Google Search Engine for this purpose<sup>2</sup>:

1. I infer that currently the search is based on your complete digit string and complete digit substrings in a web site, rather than on parsing your input float string and numeric web substrings into floats then comparing those with a tolerance or on returning results for which your input string matches a proper leading substring of a web string or vice versa. This has several consequences:
  - (a) You will probably obtain disjointly different sets of web sites for different numbers of entered digits.
  - (b) Entering too few digits is likely to overwhelm you with irrelevant elementary mathematics tutorials, serial numbers, etc.
  - (c) Entering too many digits is likely to return no result.

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<sup>2</sup>The Microsoft Edge and Apple Safari browsers currently allows a choice of Google, Bing, Yahoo or DuckDuckGo as a search engine .

- (d) Most computed floats published since the internet are computed using IEEE binary64 arithmetic, which is almost 16 digits, the last one or more of which are often incorrect due to discretization, rounding errors and catastrophic cancellation. Authors who realize this should round or at least truncate their web numbers to at most the number of digits they believe are almost certainly correct, but realization of that is not a requirement for web publication. Thus if your number was computed with IEEE binary 64, a reasonable strategy is to first try all 16 digits, then 15, 14, etc. down to where you obtain promising results or it seems likely that you will not obtain any. When doing so, remember that an author might have rounded or truncated a number on their web page, so you should try both if they differ. When rounding, always do so from your complete original number because, for example, rounding 1.546 to 1.55, then subsequently rounding that to 1.6 is different than rounding 1.546 directly to 1.5.
  - (e) But numbers of interest to mathematicians are often computed to more than 16 digits, so if the next thing to try if your number was computed to more than 16 digits is to try 17 digits, then 18, etc. Some authors tend to prefer a multiple of 3, 4 or 5 for either the number of fractional digits or the number of fractional digits – particularly if they group digits with spaces or commas for legibility, which might affect the success of the web search.
2. Decimal fraction notation seems more successful than scientific notation, perhaps partly because of variations in spacing, characters, raised exponents, and programming notations such as using E or e for “ $\times 10^{\wedge}$ ”. However, if your float has magnitude that is quite large or small compared to 1, then also try omitting the power of 10 from scientific notation because the search engine might terminate a lexical scan when it encounters any character other than a digit or decimal point in a web page.
  3. For negative numbers, try both the negative entry and its absolute value.
  4. For numbers having magnitude less than 1, try both with and without a leading 0 digit before the decimal point.
  5. If a web page containing a promising match includes a name for a constant (such as “Artin’s constant”), then branch into a search using that clue.

Somewhat different tips might be appropriate for other search engines, and Google’s search algorithm might evolve to make some of the tips here unnecessary. To learn, experiment with float approximations to a few known mathematical constants that are not too obscure or too well known. For example, try

0.91596559.

## 5 Free web applications

If you have internet access as you read this, then you might want to try each of the web applications in this section as you read the subsections. With good eyesight, most of them are usable even with most smart phones.

To become acquainted with them, using simple then less simple nonfloat constants from books and articles, try entering their floating-point approximations of varying precisions. Also try random floats having differing numbers of significant digits. As the number of significant digits increases, random floats are decreasingly likely to be the truncated or rounded value of any nonfloat expression simple enough for any of these programs to propose in a reasonable amount of time. An application is **generous** if it often suggests expressions for such random-number entries, versus **parsimonious** if it rarely does. It is alright for an application to be generous, but it is important for you to realize it. Generosity can be helpful for determining bounds or efficient sufficiently accurate approximations or for suggesting the dominant terms of a longer exact result.

“Perfection is the enemy of success”  
– Mark Cuban

Every float is exactly representable as an easily computed rational number, but floats cannot exactly represent all rational numbers, and floats are often approximations to irrational numbers. If your float approaches a limit as the working precision of the calculation approaches infinity and that limit is expressible as a nonfloat expression, then that limit is what you want. **Perfect agreement with a mere approximation prevents that desired result.**

### 5.1 The On-line Encyclopedia of Integer Sequences

One of the main purposes of the OEIS site at <https://oeis.org/> is to identify integer sequences such as proposing that the truncated sequence 8, 16, 32, 64, . . . might be a subsequence of the sequence whose  $n$ th element is  $2^n$ . Another goal of this database is to provide accurate and comprehensive information about these sequences.

The successive digits of a decimal fraction can be regarded as an integer sequence, and an under-advertised capability of OEIS is to propose nonfloat constant expressions that closely approximate your float. OEIS has sequences of up through about 100 digits or more for about 10,000 constants regarded as important by the many contributors who posted them. To attempt identifying nonfloat candidates for your float, you can enter a comma-separated list of some significant digits of the *absolute value* of its decimal fraction value, such as 3, 1, 4, 1, 5, 9, 2, 6 . OEIS then attempts to match your entered successive digits with *any* subsequence of its sequences. It then lists information about each of its integer sequences that contain your sequence as a subsequence. However, it is better to enter your float digits as a positive decimal fraction with a decimal point, such as 3.1415926. (Do not use scientific notation.)

Any matched table entries can contain hyperlinks to references and contain computer-algebra program fragments to compute the constant to any number of digits efficiently. There is also usually a Keyword link labeled **cons** that leads to the precomputed value

of the constant up to a median of about 100 significant digits. The references sometimes contain values with thousands of digits.

This exact-match *decimal radix search* requires you to *truncate* trailing digits of the absolute value of your float *rather than round*. However, the hardware or software that computed your float is approximate and almost certainly used rounding rather than truncation. Therefore you should truncate enough of your trailing digits to be reasonably confident that your submission matches exactly the leading digits at OEIS.

If your computed float ends with a sequence of digits 9, then also try what you get by adding 1 to the last digit. If your computed float ends with a sequence of digits 0, then also try what you get by subtracting 1 from the last digit, because those minimal relative changes to your approximate float can dramatically alter the digit sequence.

If you enter too few digits then you will obtain numerous matches, most or all of which are irrelevant. However, with about  $10^5$  numbers in the data base, entering 8 or more significant digits is usually sufficient to achieve at most one match, and almost all of the OEIS sequences for floats contain at least that many digits. Therefore you can usually truncate to about 8 significant digits without much chance of false matches or of entering more digits than are in a table entry. However, some of the table entries are there because they are surprisingly close to other constants of widespread interest, such as  $\pi$ . Therefore you should always compare *all* of your digits with corresponding ones stored at the `cons` link in the table.

Moreover, if you do obtain more than one match, then it is wise to do this comparison for all of them or to enter more correctly truncated digits if possible to reduce the match count to a comfortably inspectable number.

Since OEIS ignores the position of the decimal point, entering 31.41592 also returns  $\pi$  rather than  $10\pi$ . It is your responsibility to notice this and to multiply the proposed exact constant by the appropriate power of 10. This base-ten **significand aliasing** increases the number of floats that can be matched.

Since OEIS can match any *subsequence* of their digit sequences, you can also exploit fractional part aliasing by entering only the fractional part of the absolute value of your float. For example, if your truncated wild float is 9.141592654, then entering it produces no result, whereas entering 0.141592654 produces  $\pi$  and gives its float value, enabling you to recognize that your wild float is probably  $6 + \pi$ . Thus it is worth trying all of your float magnitude, then only the integer part of that magnitude if different.

Beware that fractional part aliasing suffers catastrophic cancellation when the absolute value of the fractional part is small compared to the absolute value of the float, whereas significand aliasing suffers no loss of precision. Therefore it is best if 8 or more significant digits remain after omitting the integer part and truncating dubious trailing digits.

Other than you exploiting these types of aliasing, OEIS makes no effort to identify more general transformations of its constants, such as  $2/3$  of a named constant or  $2/3$  plus a named constant.

All OEIS integer sequences and float digit sequences are assigned an OEIS name consisting of the letter “A” followed by six digits. Although such names are difficult to remember and do not acknowledge the discoverers, they are unique and serve as a *de facto* standard – particularly when there is no other widely recognized name. In contrast,



traditional names can have various spellings, subsets, and orderings of all the discoverers, as well as values differing by a factor of 2, etc, on account of one author using a radius where another uses a diameter, for example.

A complimentary way to use OEIS for named constants is to enter a name, such as *grazing goat*, or an OEIS name such as A133731. If successful, you will find the same information as for entering the truncated float value 1.158728473 .

Brief sorted descriptions of all the constants together with links to their pages are at

[https://oeis.org/wiki/Index\\_to\\_constants#Description](https://oeis.org/wiki/Index_to_constants#Description)

Some of these constants are tabulated, but most are isolated published constants, drawn from many areas by many contributors.

Even if another tool has already returned a satisfying result, OEIS and Steven Finch's books are good places to seek more information about a constant.

## 5.2 Ordner: index of real numbers

Fredrik Johansson [11] posted an online table of over 3000 sign-aliased real numbers, many of which I have not found elsewhere. To avoid a tedious amount of scrolling, there is a clickable index of intervals that transfer you to the subtable bracketing the absolute value of your float. For each subtable float there are one or more links to alternative equivalent expressions having that 30-digit value to within one unit in the last place.

As a bonus, a link to the Fungrim home page reveals a table of contents with links into a collection of useful function identities, some of which I have not seen elsewhere.

## 5.3 WolframAlpha

If you enter a float into WolframAlpha at <https://www.wolframalpha.com/>, the results usually include up to three guesses for a nonfloat constant that the float approximates.

A way to test the tools described in this article is to approximate a nonfloat constant to a generous number of significant digits, then see how many of them are necessary to produce an equivalent expression, if any. Accordingly, I approximated

$$1 - \frac{3\sqrt{3}}{1 + \sqrt{3} + \pi} \tag{1}$$

to 18 digits, then tried entering that and successively fewer digits into WolframAlpha. Table 2 lists the results for floats *rounded* to 10 and to 11 significant digits, together with entries having 11 correct digits followed by one and by two incorrect digits. Underlined digits in the entries differ from the exact expression by at most 0.5 units in the last place.

Table 2: WolframAlpha test for identifying  $1 - \frac{3\sqrt{3}}{1 + \sqrt{3} + \pi}$  from a float approximation

Entry	digits	Result (Agrees with input to 0.5 in last bold digit) $\equiv$
0. <u>1153442566</u>	10	$\frac{11 - 7\pi + 2\pi^2}{\pi(21 + \pi)} \approx \mathbf{0.11534425649366}$
		$\frac{1}{\text{root of } x^4 - 75x^2 + 10x - 99 \text{ near } x = 8.6697} \approx \mathbf{0.11534425656239}$
		$\text{root of } 99x^4 - 10x^3 + 75x^2 - 1 \text{ near } x = 0.115344 \approx \mathbf{0.11534425656239}$
0. <u>11534425658</u>	11	$\frac{1 - 2\sqrt{3} + \pi}{1 + \sqrt{3} + \pi} \approx \mathbf{0.115344256584835}$ ✓
		$\frac{-2e e! + 124 - 35e + 5e^2}{136e} \approx \mathbf{0.115344256579706}$
		$\text{root of } 92x^4 + 49x^3 - 27x^2 + 37x - 4 \text{ near } x = 0.115344 \approx \mathbf{0.1153442565821563}$
0. <u>115344256580</u>	12	$\frac{1 - 2\sqrt{3} + \pi}{1 + \sqrt{3} + \pi} \approx \mathbf{0.1153442565814835}$ ✓
		$\frac{2(15\sigma_S - 1)}{5(6\sigma_S + 73)} \approx \mathbf{0.1153442565818196}$ where $\sigma_S$ is Somos's quadratic recurrence constant
		$\pi (\text{root of } 100x^4 - 152x^3 - x^2 - 27x + 1 \text{ near } x = 0.0367) \approx \mathbf{0.1153442565828298}$
0. <u>1153442565800</u>	13	$\frac{45419\pi}{1237062} \approx \mathbf{0.1153442565800221712}$
		$\frac{3e e! + 295 - 86e - 9e^2}{94e} \approx \mathbf{0.15344256579947}$
		$\frac{1}{45} (4C + 30 - 11\pi - 40\pi \log(2) + 27\pi \log(3))$ $\mathbf{0.115344256580078601}$ , where $C$ is Catalan's constant

Notice that:

1. Alpha did not propose an equivalent for 10 correct digits.
2. Alpha proposed an equivalent expression for 11 or more correct digits provided they were followed by no more than 1 incorrect digit.
3. Thus Alpha almost always finds expressions that differ from your input by at most a few units in the last place that you entered, making those differences not a good criteria for favoring one alternative over the others.
4. The returned candidates are usually nonequivalent, so Alpha is generous.

So with typically three nonequivalent expressions all having about the same differences from your input, how are we to know which if any are the limit we seek?

By the principle of parsimony, you should prefer the least complex, but the three alternatives for 11 correctly-rounded digits don't visually differ much in complexity. Here is a way to gain some evidence: If you can compute your wild float to more digits, perhaps using Alpha, then do so and round the result enough to be reasonably confident that the last digit or at least the one before it is correctly rounded, then enter that float. Impostors for the lower-precision input are less likely to appear for the higher-precision float. For example, the correctly rounded 18-digit value for expression (1) is 0.115344256581483524, and entering that returns the three alternatives

$$\begin{aligned} \frac{1 - 2\sqrt{3} + \pi}{1 + \sqrt{3} + \pi} &\approx \mathbf{0.115344256581483524414074}, \\ \frac{26534199\pi}{722703038} &\approx \mathbf{0.1153442565814835251297}, \\ \frac{-678 + 46e + 97e^2}{2(-10 + 167e + 36e^2)} &\approx \mathbf{0.1153442565814835236159}. \end{aligned}$$

Notice that the first nonfloat is the same as for entering 11 or 12 correctly rounded digits in Table 2. The other two nonfloats for 11 and 12 digits have been replaced with other expressions and were therefore almost certainly impostors. It is possible that the true limit would not appear until more than 18-digit precision, making the first entry for 18 digits an impostor too that would disappear at higher precision. However, it is typical of true limits that once they appear at a certain precision, they will appear for all higher precisions. Therefore it is advisable to enter increasing numbers of probably-correctly rounded digits until one of the alternatives convincingly persists or you become convinced that Alpha does not model your float. If two or more candidates persist, then perhaps they are equivalent.

If you cannot compute more digits for your original float, then try rounding it to fewer digits to see which alternatives persist in that direction.

When Alpha returns more than one result and only one has dramatically less complexity than the others, there is some justification for preferring it. However, I prefer to **always try at least two different precisions to assess transience versus persistence.**

Alpha mostly uses the PSLQ **integer relation** algorithm<sup>3</sup> [7, 8]: Given a basis vector  $\mathbf{c} = [c_1, c_2, \dots]$  of nonzero symbolic and/or float constants, the algorithm returns either an indication of failure or a minimal 2-norm vector of integers  $[n_1, n_2, \dots]$  not all 0 such that the inner product  $n_1c_1 + n_2c_2 + \dots \simeq 0.0$  and  $\gcd(n_1, n_2, \dots) = 1$ .

1. One way to exploit this algorithm is to make  $c_1$  be the given float  $\tilde{x}$  and choose commonly occurring nonfloat constants such as 1,  $\pi$ ,  $\sqrt{2}$ , and  $\ln 2$  as the other constants. If the algorithm returns a vector  $[n_1, n_2, \dots]$ , then a close-fitting candidate is the *rational linear combination*

$$\tilde{x} \simeq -\frac{n_2}{n_1}c_2 - \frac{n_3}{n_1}c_3 - \dots .$$

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<sup>3</sup>The LLL algorithm can also be used for this purpose.

The last entry in Table 2 is of this type.

2. Another way to exploit this algorithm is if

$$(n_1c_1 + n_2c_2 + \dots) \tilde{x} + (n_kc_k + n_{k+1}c_{k+1} + \dots) \simeq 0.0,$$

then

$$\tilde{x} \simeq - \frac{n_1c_1 + n_2c_2 + \dots}{n_kc_k + n_{k+1}c_{k+1} + \dots}.$$

This *linear fractional* category of model was used for the correct identifications and some of the incorrect ones in Table 2.

3. Another way to exploit this algorithm is if

$$n_1 + n_2\tilde{x} + \dots + n_{m+1}\tilde{x}^m \simeq 0.0 \tag{2}$$

then  $\tilde{x}$  is approximately one of the zeros of the polynomial  $n_1 + n_2x + \dots + n_{m+1}x^m$ . This can be tried for  $m = 1, 2, \dots$  up to some maximum degree that increases with the number of significant digits. If PSLQ returns a vector of integers, then an approximate polynomial zero algorithm can then be used to determine which zero is closest to the given float. Several alternatives in Table 2 are such an *algebraic number* candidates.

4. The minimal polynomial integer relation can be generalized to include predetermined exact irrational constants with one or more powers of  $\tilde{x}$ , such as

$$n_1\pi + n_2\tilde{x} + n_3\tilde{x}^2 \simeq 0.0.$$

For degree 2 models such as this, the quadratic formula could be used to express a successful PSLQ candidate result explicitly in terms of  $\pi$ , rather than implicitly using Root [...].

5. For a power product model of the form  $\tilde{x} \simeq b_1^{r_1}b_2^{r_2}\dots$  with given positive non-float base constants  $b_k$  and unknown rational exponents  $r_k$ , if  $n_1 \ln \tilde{x} + n_2 \ln b_1 + n_3 \ln b_2 \dots \simeq 0.0$ , then

$$\tilde{x} \simeq \exp\left(-\frac{n_2 \ln b_1 + n_3 \ln b_2 + \dots}{n_1}\right) = b_1^{-n_2/n_1}b_2^{-n_3/n_1}\dots.$$

The irrational factors of an irrational exponent can be included in the bases to make this category fit a model such as  $\tilde{x} \simeq 2^{r_1}3^{r_2\sqrt{5}}e^{r_3\pi} = 2^{r_1}\left(3\sqrt{5}\right)^{r_2}(e^\pi)^{r_3}$ .

6. For a model of the form  $f(\dots)$ , where  $f$  is invertible with respect to at least one of its arguments and that argument model is one of the above types or recursively of this functional type:

- Using predetermined constants for any other arguments, apply an inverse of  $f$  to the given float  $\tilde{x}$  giving  $\tilde{y}$ .

- If the argument model is successfully fit to  $\tilde{y}$ , giving a nonfloat candidate constant  $y$ , then return  $f(y)$ .

The computing time grows rapidly with the number of elements  $m$  in the basis vector and the number of words in the float significands. Moreover, the PSLQ algorithm can require the input float to have  $mn$  significant digits of precision to return a correct  $m$  component vector having integer components of up through  $n$  digits even if many of these integer coefficients have much smaller magnitudes, including 0. Therefore, although specialized efforts to identify particular floats have used thousands of digits and hundreds of vector components, the general-purpose programs described here use mostly basis vectors having  $\lesssim 8$  elements.

### More advise for using Alpha to identify floats:

1. If you enter 18 or more significant digits, including any trailing 0 digits, then Alpha uses arbitrary precision rather than 16-digit machine floats. Arbitrary-precision floats are slower but more than 18 digits are necessary for PSLQ to determine many published constants.
2. Alternatively, you can append a suffix of the form ‘ $n$  to the significand of your input to indicate that your best guess for the Precision is the value of  $n$ , which can be a decimal fraction. For example, if your input for the example in Table 2 was computed by iterative equation solving using 16-digit IEEE binary64 hardware and you guess that most probably about 14 digits are correctly rounded, then you could start by entering 0.1153442565814834‘14 . This *strongly recommended* alternative has the advantage that you can enter *all* of the computed digits, which which are more likely to be correct than the value rounded or truncated to 14 digits, but without having Alpha assume that all but perhaps the last one are correct. In the other direction, if your computed 16-digit machine float is 0.1615000000000000 and you are rather confident that all are correct, then you can enter it more concisely and reliably as 0.1615‘16 . Different from the apostrophe, the accent grave character ‘ is near the upper left corner on many keyboards and might look different on your keyboard or screen.
3. Software does not always display all of the digits in its significands, and internet browsers paste into web applications such as Alpha only the *characters* that you highlight when copying. For example, *Mathematica* displays by default only 6 of the approximately 16 significant digits in the widespread IEEE binary64 machine floats, and displays none of the 8 or more stored *guard* digits in arbitrary-precision floats. Often some of those hardware or guard digits are correct and therefore helpful to try entering. Therefore, learn how to display all of the digits in your software’s floats.<sup>4</sup>

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<sup>4</sup>One way to do this in *Mathematica* is to copy all of the displayed digits then paste into a *notebook*, then copy all of that fully-displayed InputForm float into Alpha. Another way is to copy the result of InputForm [float] into Alpha.

4. Unless you are an exceptionally fast accurate digit typist, highlight then **copy the entire float including its hidden digits rather than type it**.
5. Then attach to the significand a suffix of the form ‘ $n$ ’, where  $n$  is your best guess for the number of significant digits that are correctly rounded. Keep in mind that cancellation and rounding errors often lose one or more digits from the working precision – and that discretization errors such as from quadrature often lose more. Interval arithmetic can provide an upper bound on cancellation and rounding errors, but not necessarily on discretization errors. Significance arithmetic such as *Mathematica*’s arbitrary-precision float arithmetic provides cancellation and rounding error estimates that are occasionally underestimates but more frequently overestimates.
6. PSLQ overfits often contain many digits in their rational numbers, such as the first alternative in Table 2 for 13 digits. Information theory and my experience suggests that results are dubious if they contain a total of more instances of operators, functions and digits than about 6 less than your number of estimated correct significant digits in your float input. Also, several terms or factors containing only simple rational numbers is less suspicious than one term containing a very complicated rational number.

## 5.4 Inverse Symbolic Calculator

Inverse Symbolic Calculator (ISC) implemented by Simon Plouffe working with Jonathon and Peter Borwein and others is currently hosted at

<http://wayback.cecm.sfu.ca/projects/ISC/ISCmain.html> .

Plouffe [14] describes its origins.

ISC provides four different techniques to apply to the float that you enter:

### 5.4.1 Simple Lookup and Browser

This technique searches a precomputed table of sorted **truncated** 16-digit base-ten **significands** of the **absolute values** of the floats, each paired with a character-string description of the constant.

That table has about 30 million pairs that are a mixture of published and tabulated constants. The sorted significands and the corresponding nonfloats are subdivided into about 9000 files whose names include some leading digits of all the significands in that file, with each file containing all of the significands having those leading digits. The search begins with a binary search on the file names in RAM, followed by a search in the appropriate file to perfectly match or bracket the significand of the float that you entered, followed by retrieval of close neighbors of those entries. Here are some **input tips**:

- ISC Simple Lookup defines a **match** as **all** of the entered digits of the significand being **identical** to that leading portion of a tabulated significand, but ignoring any entered digits beyond 16.
- The last one or more digits of your computed float are often not the correctly truncated ones. To maximize your chance of success, try to compute as many digits as is practical to be reasonably confident that the first 16 digits are correctly truncated, then enter at least those 16 digits. Your float was probably computed with software that rounds rather than truncates, so at least truncate the last computed digit.
- If you cannot compute more than 16 digits, then truncate to as many as your best guess for how many are correctly truncated digits of the limit you seek.
- If the least significant truncated digit is a 9, then also try truncating what you get by adding 1 to that 9 digit. If the least significant truncated digit is instead a 0, then also try truncating what you get by subtracting 1 from that 0 digit. Those minimal relative changes to your entered float can dramatically alter the trailing digit sequence.
- You can enter as few as 5 significant digits, but with about 30 million entries having significands 0.1000000000000000 through 0.9999999999999999, the mean distance between table significands is

$$\frac{0.9999999999999999 - 0.1000000000000000}{30 \times 10^6} \simeq 3.3 \times 10^{-8}.$$

Thus you can expect an unwelcome number of matches if you enter fewer than seven or eight significant digits.

- You are offered the opportunity to browse nearby table entries even if there are no matches. Doing so might reveal one or more candidate nonfloats that agree closely and are plausible considering your knowledge of the problem domain. Browse even if there are matches, because your input might not be as correctly truncated as you assumed.
- The table contains many instances of nonfloats that all have the same 16-digit significand value. The corresponding nonfloats are often but not always equivalent to each other within a sign and a factor that is a power of 10. Approximating these nonfloats to higher precision can support such aliased equivalence or reveal the lack of such equivalence.
- Moreover, if you can compute your wild float to more than 16 digits and you have software that can evaluate a table nonfloat to more than 16 digits, then doing so permits you to determine how many more digits of that nonfloat agree with your higher-precision wild float. Beware though that the table nonfloats are character strings some of which are Maple syntax or *Mathematica* syntax or neither, so you

might have to type a syntactically appropriate translation into your software rather than merely copy and paste.<sup>5</sup>

- If you do not obtain at least one promising result, then also try entering fewer and more truncated digits up through 16, covering what you guess spans most of the possible actual correctly truncated digits of your wild float.

As a test example, the correctly truncated 16-digit value of

$$-10\sqrt{\frac{2}{37}(8\sqrt{3}-9)} = -5.123557917376186. \quad (3)$$

Table 3 summarizes the results of entering various further truncated and rounded leading digits to expose the consequences of rounding rather than truncating – and of appending incorrectly truncated digits beyond the sixteenth.

Table 3: ISC Simple Lookup test results for approximated  $\left| -10\sqrt{\frac{2}{7}(8\sqrt{3}-9)} \right|$

Input	digits	Treatment	Result	?
5.1235	5	<b>truncated</b>	<b>38 matches</b> include $12^{1/4}/\sqrt{8+9^{3/4}}$	Too many
5.1236		rounded	<b>360 matches.</b> They <b>don't</b> include $12^{1/4}/\sqrt{8+9^{3/4}}$	Too many
5.123557	7	<b>truncated</b>	Matches only $12^{1/4}/\sqrt{8+9^{3/4}}$	✓+
5.123558		rounded	6 matches that <b>do not</b> include nearby $12^{1/4}/\sqrt{8+9^{3/4}}$	All bad
5.1235579173	11	<b>truncated</b>	Matches only $12^{1/4}/\sqrt{8+9^{3/4}}$	✓+
5.1235579174		rounded	No match, but browser lists adjacent $12^{1/4}/\sqrt{8+9^{3/4}}$ , matching 10 digits	✓
5.12355791730	12	incorrect last digit	No match, but browser lists adjacent $12^{1/4}/\sqrt{8+9^{3/4}}$ , matching 10 digits	✓
5.12355791737618	15	<b>truncated</b>	Matches only $12^{1/4}/\sqrt{8+9^{3/4}}$	✓+
5.12355791737619		rounded	No match, but browser lists adjacent $12^{1/4}/\sqrt{8+9^{3/4}}$ , matching 14 digits	✓
5.1235579173761861	17	<b>incorrect last digit</b>	Matches only $12^{1/4}/\sqrt{8+9^{3/4}}$	✓+
5.1235579173761869		<b>incorrect last digit</b>	Matches only $12^{1/4}/\sqrt{8+9^{3/4}}$	✓+

<sup>5</sup>Or, try typing “N [nonfloat, desiredPrecision]” into WolframAlpha, where *nonfloat* is the string copied and pasted from ISC. WolframAlpha is somewhat tolerant of varying mathematical notation.



For each match, ISC lists the truncated 16-digit base-10 significand without a decimal point, equated to a corresponding nonfloat, such as

$$512355791737618 = 12 \wedge (1/4)/(8 + 9 \wedge (3/4)) \wedge (1/2). \quad (4)$$

To process this match:

1. The ratio of your input 5.123557917376186 to the significand 0.5123557917376186 is 10, so multiply the candidate nonfloat by 10 to de-alias the significand.
2. Then negate that, because our float of interest in equation (3) is negative and we necessarily entered its absolute value, giving our final result

$$-10 \frac{12^{1/4}}{\sqrt{8 + 9^{3/4}}}. \quad (5)$$

3. For verification of our de-aliasing and of our interpretation of the table nonfloat, we should approximate expression (5) to preferably more than 16 digits, then truncate to trusted digits, giving -5.123557917376186, which agrees with the right side of equation (3) to 16 digits.
4. It is not difficult to transform the de-aliased nonfloat candidate (5) into our test expression on the left side of (3). But what if our input was a wild constant that happens to agree with an impostor to 16 digits?

**For candidates resulting from tables, if you can compute your float input and a candidate nonfloat to higher precision than used in the table, then it is wise to assess exact and near-exact matches to higher precision than used in the table to help expose impostors or gain confidence in the correctness of the nonfloat.**

5. However, beware that the table nonfloats are character strings some of which are Maple or *Mathematica* syntax or neither. Therefore you might have to type a syntactically appropriate translation into your software rather than merely copy and paste.<sup>6</sup>

Beware also that the table was assembled over many years from many computed and printed sources that used varying syntactic conventions. For example:

- Gam, Gamma, GAM and GAMMA all denote the Gamma function, whereas gamma denotes the Euler-Mascheroni constant.
- sqrt and sr both denote the principal square root.
- A polynomial in  $x$  or root(*a\_polynomial\_in\_x*) denotes the zero of that polynomial nearest the given float.

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<sup>6</sup>Or, try exploiting Alpha's heuristic expression parser by typing "N [*nonfloat*, *desiredPrecision*]", where *nonfloat* is the string copied and pasted from ISC.

- Non-real expressions often implicitly denote the real parts of those expressions.
- As with our regrettably ambiguous traditional mathematics notation, sometimes  $16^{3/4}$  means  $16^{(3/4)}$  rather than  $(16^3)/4$ ,  $\sin e + 3$  means  $\sin(e + 3)$  rather than  $\sin(e) + 3$ , and  $7/3\pi$  means  $7/(3\pi)$  rather than  $(7/3)\pi$ , etc.

**Step 3 above enables you to verify or refute your interpretation of the nonfloat syntax, then to try another interpretation if necessary.**

Some nonfloats in the table contain either a common name constant such as Bernstein or a red OEIS web link labeled with an OEIS constant having a name of the form “A” with a six-digit suffix. Either way, you can search for their definitions at <https://oeis.org/>, as described in subsection 5.1.

### 5.4.2 Smart Lookup

This technique uses the same lookup table about 100 times, but precedes each lookup with a transformation of your float. For example, smart lookup might add  $3/7$  or multiply by  $3/7$  or apply  $\ln(\dots)$ . For a given number of input digits, the average number of matches is thus about 100 times as many as for simple lookup. However, some transformations lose precision. Therefore smart lookup requires you to enter at least 10 digits, and the transformation of your input is done using 26-digit significands. Thus it is beneficial to enter floats having up through 26 correctly rounded digits even though the forward table floats have only 16-digit significands.

For example, computed to 26 significant digits,

$$-\frac{1}{4} - 10 \frac{12^{1/4}}{\sqrt{8 + 9^{3/4}}} \simeq -5.3735579173761866715453232. \quad (6)$$

Applying simple lookup to the absolute value of this float truncated to 16 digits returns no match, and the ISC browser lists no result that agrees to more than 5 digits. However, applying *smart lookup* to the absolute value of the right side of equation (6), computed with arithmetic that rounds, returns the table

<i>Function</i>	<i>Result</i>	<i>Precision</i>	<i>Matches</i>
$K - 1/4$	5.1235579173761866715453232	16	1

The *Function* entry,  $K - 1/4$  is a transformation that was applied to your float, with  $K$  representing both the nonfloat that you seek and its approximate value

$$5.3735579173761866715453232.$$

The *Result* entry is the result of applying the transformation to your float as computed with 26-digit significands. The *Precision* entry is the number of leading digits in the table that matched those in your transformed float. The *Matches* entry is the number of matches for that transformation.

If you click on the *Function* entry  $K - 1/4$ , then ISC shows the result of doing *simple lookup* with a truncated significand of the transformed *Result* float, which gives

$$\frac{12^{1/4}}{\sqrt{8 + 9^{3/4}}}, \quad (7)$$

together with an opportunity to browse around it.

If you want to further investigate one of these nonfloats that have agreement acceptable to you, then, for example, you should

1. Numerically approximate the nonfloat, preferably to at least 16 digits, giving

$$K = 12^{1/4} / \sqrt{8 + 9^{3/4}} \simeq 0.512355791737618667.$$

2. From the ratio of that float to the float in the Result entry, determine a power of 10 by which to multiply the nonfloat:

$$\frac{5.1235579173761866715453232}{0.512355791737618667} = 10.0000000000000000.$$

3. Multiply the table nonfloat (7) by that power of 10 giving the magnitude de-aliased nonfloat

$$y = 10 \frac{12^{1/4}}{\sqrt{8 + 9^{3/4}}}$$

4. Solve  $y = K - 1/4$  for K either manually or by computer algebra :), giving

$$K = y + 1/4 = 10 \frac{12^{1/4}}{\sqrt{8 + 9^{3/4}}} + \frac{1}{4}$$

5. Negate this result to de-alias the sign, giving the nonfloat candidate for our float of interest

$$- \left( \frac{1}{4} + 10 \frac{12^{1/4}}{\sqrt{8 + 9^{3/4}}} \right).$$

To summarize:

1. We made the input float positive by negating it.
2. Smart lookup found a transformation that lead to a matching significand in the Simple lookup table.
3. We multiplied the corresponding nonfloat  $y_1$  in the Simple lookup table by an appropriate power of 10 to make the value of the resulting nonfloat  $y_2$  match the *Result* value in the Smart lookup table.
4. We then applied the inverse of the *Function* of K in the Smart lookup table to  $y_2$  giving  $y_3$ .
5. We then negated  $y_3$  giving the candidate for our original float.

More generally there might be several rows in the returned table for several different transformations that resulted in matches. (The number of rows tends to decrease as you increase the number of entered digits above the minimum required 10 digits.)

Click on each of the Function entries to see the corresponding nonfloats and optionally browse their neighbors.

For this example, entering fewer than 16 digits produced no smart lookup result.

It is worth starting with Simple Lookup first because it is faster, it does not require an inverse transformation, and it permits browsing the neighborhood of your input even if there are no matches.

### 5.4.3 ISC Integer Relation Algorithms

ISC provides an Integer Relation alternative that you should try too even if simple or smart lookup gives a promising result. Integer Relations requires inputs of 16 through 32 significant digits. Integer Relation checks for algebraic numbers through fifth degree and for rational linear combinations of the basis vectors

$$\begin{aligned} & [e, \pi, \gamma, \text{Ei}(1), W(1), 1], \\ & [\sqrt{3}\pi, \ln 3, \ln 2, \gamma, \sqrt{2}\pi], \\ & [\pi^2, \text{Catalan}, \pi \ln 2, \sqrt{2}\pi^2, (\ln 2)^2], \\ & [\pi^3, \zeta(3), \pi^2 \ln 2, (\ln 2)^3, \sqrt{3}\pi^3, \sqrt{2}\pi^3]. \end{aligned}$$

These basis vectors might seem rather few and specialized. However, a noticeable portion of wild constants in the printed tables of Section 2 are representable with a subset of at least one of these vectors or with algebraic numbers of degree  $\leq 5$ .

It is wise to compute as many digits as you can up through 32. There is no need to truncate, and it is OK for a few of the last digits to be incorrectly rounded.

### 5.4.4 Generalized Expansions

You must enter at least 16 significant digits for this option, and it is most appropriate to *round* your approximate float to where you are reasonably confident that the last digit is correctly rounded.

This option computes a truncated continued fraction from your input float, and computes similar truncated infinite representations such as a truncated infinite product and a truncated Egyptian fraction. This can be helpful because these alternatives might suggest patterns that might be useful for computing the float to arbitrarily many digits more efficiently than the method you used. Moreover, the GFUN package [19] is then applied to these truncated representations to try guessing a generating function for the infinite representation. (It is for you to prove that it *is* such a generating function.)

Also, you can copy the comma-separated sequence of integers for one of these truncated representations, then paste it at <https://oeis.org/> to attempt learning more about it, possibly including a closed form. The [Hints] button on the OEIS home page invokes advice that includes:

“Enter about 6 terms, starting with the second term. Leave off the first term or two, because people may disagree about where the sequence begins. Don’t enter too many terms, because you may have more terms than are in the OEIS data base.”

Generalized Expansions and two related tools GCF.TXT by Dougherty-Bliss and Zeilberger [6] and the Ramanujan Machine by a team of nine authors [16] are intriguing but adjacent to the topic of this article, so they are not described further here.

## 5.5 Inverse Symbolic Calculator Plus

Inverse Symbolic Calculator Plus at <https://isc.carma.newcastle.edu.au/> was a different interface to essentially the same smart lookup and integer-relations models as ISC, ISC+ has had the message “down for maintenance indefinitely” from late 2018 through at least January 2022. However, as of this writing there is a link there titled “The original ISC” that links to the ISC site already discussed in subsection 5.4.

## 6 Freely downloadable standalone applications

Most of the implemented integer relation models and precomputed table-lookup entries of the software described in this article are based on one implementer’s perception of what patterns of nonfloat constant expressions are most likely to fit the combined needs of the intended users. There is a consequent efficiency to the extent that the implementer’s choices are appropriate for you. However, such an implementation is almost certain to miss some very concise results fitting unimplemented patterns.

The MESSearch and RIES software address this issue by trying all possible expressions composed of user-selected rational numbers, symbolic constants, operators and functions – up to a certain total complexity, time limit or memory limit. By default both programs use *bidirectional search*: The **forward table** is like the ISC table, and the **backward table** is like the ISC Smart Lookup transformations. However, rather than using *precomputed* forward and transformation tables, for each given float both applications build new tables from your selected components in a breadth-first way, interleaving the searching with this table building. The forward table begins with a selected set of simple rational numbers and named constants together with a set of arithmetic operators and functions. More complicated forward expressions are formed by applying all chosen functions and operators to existing entries in an order such that the predicted expression complexity grows approximately monotonically. The backward table starts with the given float, and applies the selected functions in an order such that the estimated resulting complexity also grows approximately monotonically. For functions and operators of more than one operand, all but one of the operands can be taken from the current forward table.

Growth alternates between the forward and backward tables so that at any one time they have roughly the same number of entries.

Even though the initial set of rational numbers is typically quite simple, arithmetic combinations of rational subexpressions can lead to larger numerator and/or denominator magnitudes.

Both programs use only hardware floating point but make the best of that limitation. Moreover, as with the 16-digit Inverse Symbolic Calculator table, if you can approximate nonfloat candidates and your wild float to greater than hardware precision, then you can assess the agreement of candidates to greater precision to help distinguish impostors from the limit you seek.

Both programs use a few simplification rules such as employing 0 and 1 identities, doing rational arithmetic, and canceling composition of a function with its inverse. However, neither program uses a computer algebra system, so sometimes the results can be usefully simplified manually or by computer algebra.

Computation ceases at your preset expression complexity limit, memory limit, time limit, or the limit of your patience when you finally interrupt a search. Both programs initially use RAM for the tables, but can resort to a much slower secondary-storage mode when allocated RAM is exhausted.

The fact that the tables are exhaustively built and searched in approximate order of increasing expression complexity has a benefit that simple candidates composed of your selected components are almost always quickly found. However, the open-ended exponential growth of the table sizes and computing time with expression complexity makes it impractical to generate expressions having complexity that is easily achievable by sparse precomputed tables such as OEIS and ISC or by integer relation models with sufficiently large-precision inputs. With a medium sized set of selected components, exhaustive search can require overnight to generate an expression containing about 12 or 13 instances of modest rational numbers, symbolic constants, arithmetic operators, and function invocations.

But the peace of mind knowing that the search was exhaustive for expressions composed of your selected rational numbers, symbolic constants, operators and functions is worth the wait. (This is a good way for you to extract value from your computer even while you sleep!)

In contrast, the other programs discussed in this article have a fixed number of table entries and/or integer relation models. The latter can accommodate expressions having any number of predetermined symbolic constants in one of several categories of forms, with arbitrarily large integers determined by PSLQ. But being non-exhaustive, those tools can and do miss some very simple results that MESSearch and/or RIES can determine.

Applying some of the transformations to the given float might produce unusable underflow overflow, or non-real results; and compositions of different backward with forward pairs might lead to identical or equivalent results. However, if the backward table has  $m$  entries and the forward table has  $n$  entries, then a useful upper bound on the number of potentially recognizable expressions is  $mn$ . If both kinds of table entries averaged the same memory space per entry, then for a given amount of memory,  $mn$  is maximized by having  $m = n$ . For example, only  $10^6$  entries in the forward table and  $10^6$  entries in the backward table gives an upper bound of recognizing  $10^{12}$  different-valued expressions.

MESSearch and RIES both offer the option of using only a forward table, but they are almost always faster and less memory consumptive bidirectionally.

## 6.1 MESSearch

MESSearch is implemented in Java and was freely downloadable from

<http://www.xuru.org/mesearch/MESSearch.asp>

That much changed site no longer lists MESSearch, but thanks to Simon Plouffe, a copy is freely downloadable from

<http://plouffe.fr/MESSearch/>

The Java Runtime Environment is often already installed on most computers, and it is freely downloadable from

<https://www.java.com/en/download/>

MESSearch uses only invertible functions or operators for the top level function or operator of its backward table, and automatically applies those inverses to the nonfloats in the forward table whose float values closely match a transformed float input.

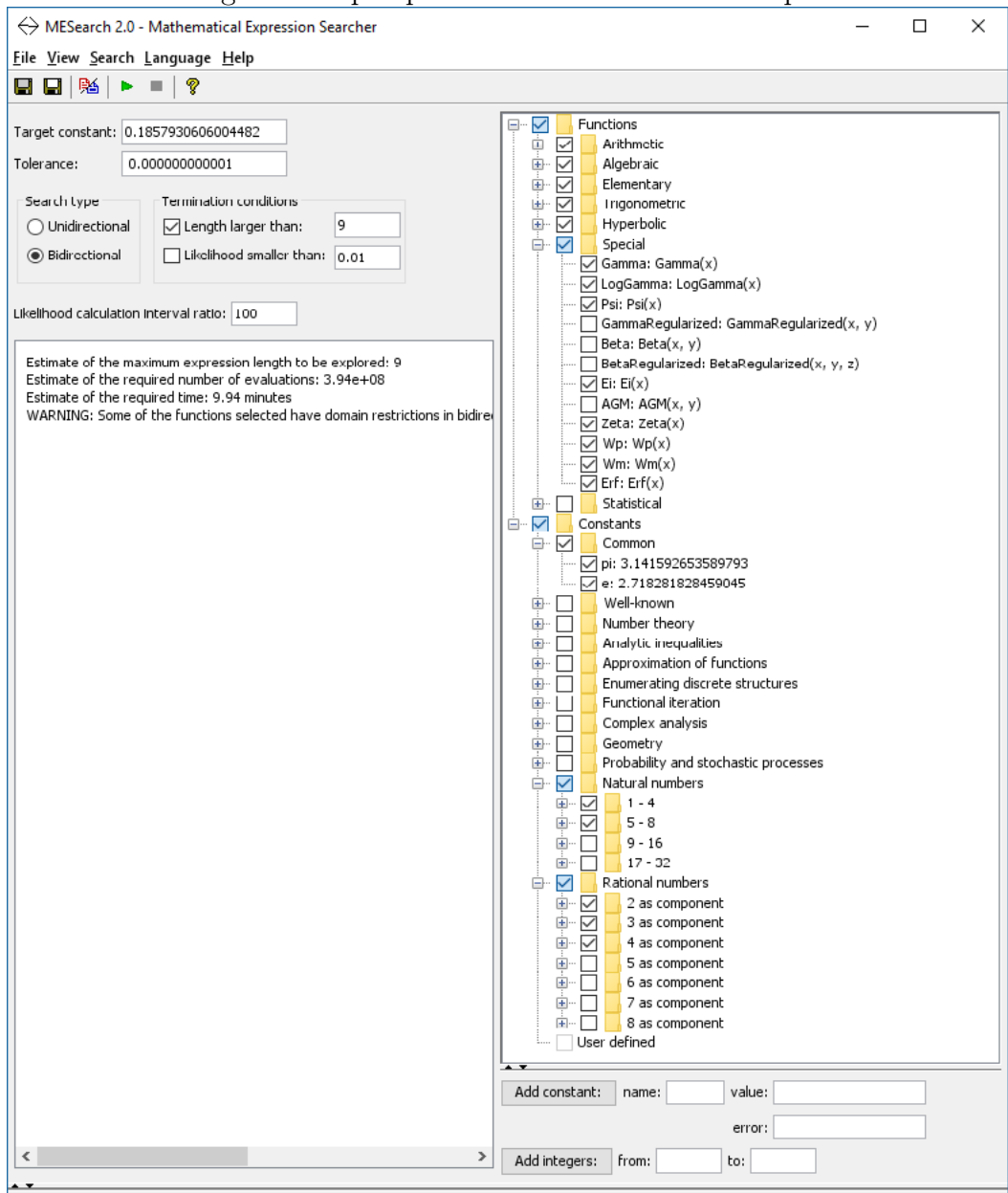
Figure 2 shows the MESSearch input panel for the test float input

$$\frac{1}{\sqrt{5 + 2\sqrt{6}} + \sqrt{5}} \approx 0.1857930606004482. \quad (8)$$

MESSearch measures expression complexity by **Length**, defined as the number of occurrences of rational numbers, named constants, functions and operators. Counting the implicit multiply, the Length of the expression on the left side of equation (8) is 12.

MESSearch offers a choice of operators, functions, named constants, natural numbers and rational numbers from which to build expressions, and Figure 2 shows my most common choices for these. Choosing only a few components permits you to search to longer expressions before exhausting memory or your patience, but it is important to include components that you expect have a nonnegligible probability of occurring in your problem domain. Functions and operators with more than one argument are particularly costly, and that is why I often initially limit those to “+”, “×”, and “^”. Memory and computing time also grow quickly with the cardinality of the initial set of named constants, natural numbers, and rational numbers. Thus I often initially select only  $\pi$  and  $e$  together with very simple natural numbers and fractions.

Figure 2: Input panel for an MESearch example



MESearch only displays candidates whose float value are within  $\pm$ Tolerance entered under the Target constant near the upper left corner of Figure 2. The default MESearch Tolerance is 1/2 unit in the last entered digit of your Target constant, which is difficult to achieve if your Target constant is all 16 digits of the hardware floats or nearly so. Consequently I often initially choose about 1000 to 10,000 units in the last entered place for Target floats having 16 significant digits, down to about 10 units in the last place for Target floats having 7 or fewer significant digits. Generous tolerances change the result likelihood estimates and are more likely to generate impostors, but unlike overly frugal tolerances, generous tolerances do not tend to lose achievable true limits. I would rather obtain a true limit together with a few impostors, then decide which are most plausible



and recompute those with smaller tolerances for more accurate likelihood estimates.

MESearch uses your entered **absolute tolerance** rather than the number of entered digits. Therefore you should enter all of your computed digits if there are less than 16 or rounded to 16 digits otherwise, even if any number of the resulting trailing digits are dubious. Therefore I entered all 16 digits of 0.1857930606004482, for which my entered absolute tolerance was  $10^{-12}$ .

Here is a summary of the most important information in the resulting output:

Length	Expression	Difference	Likely <sub>1</sub>	Likely <sub>2</sub>	Likely <sub>3</sub>
8	$((e^{\operatorname{arccosh}(5)})^{1/2} + 5^{1/2})^{-1}$	0.0	0.5103	0.9135	0.9187
9	$\operatorname{arctanh}(\cos(\tan(\sin(4/3/\sinh(\cos(4))))))$	6.769e-13	3.3e-11	0.8515	
9	$\operatorname{arctan}(\psi(2 - \tan(\operatorname{Ei}(\log\Gamma((3/4)^2)))))$	6.769e-13	3.3e-11	0.8515	

Watching the output panel evolve provides some entertainment while you wait and helps you decide whether and when to terminate the search manually.

The first expression that met the tolerance was at complexity Length 8. The three Likelihood estimates are based on the fact that the expressions are all possible expressions composed of the selected components, computed in approximately monotonic non-decreasing order of complexity. The reason for the two vacant Likely<sub>3</sub> entries is that Likely<sub>3</sub> can be computed only upon **completion** of searching its Length level, and I manually halted the search during Length 9.

Clicking a column heading sorts the rows by the values in that column, so I usually click the middle Likely<sub>2</sub>, which the implementer Salsamendi states is most often most correct.

If this input float was a wild float, then having the 0.0 Difference between the entered float and the value of the Length 8 Expression, together with its two likelihoods of greater than 90% and one of greater than 50% would lead me to strongly suspect that the corresponding expression is a correct limit. In contrast, the very low values for half the likelihoods of the two Level 9 candidates and their significantly greater Differences would lead me to strongly suspect that they are impostors.

The result subexpression  $e^{\operatorname{arccosh}(5)}$  having Length 4 is quite different from the corresponding test subexpression  $\sqrt{5 + 2\sqrt{6}}$  in equation (8). However, they are provably equivalent, and MESearch almost certainly would have generated the more recognizable equivalent

$$\left(\sqrt{5 + 2\sqrt{6}} + \sqrt{5}\right)^{-1}$$

at Level 11.

Although most people prefer algebraic numbers to be expressed using radicals or trigonometric functions if possible and practical, other representations often have lower complexity by most quickly-computed measures.<sup>7</sup>

Salsamendi [18] describes the data structures, algorithm, and time complexity.

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<sup>7</sup>Many computer algebra systems have transformation functions that can help you seek alternative representations of result candidate expressions.

## 6.2 RIES (Rilybot Inverse Equation Solver)

RIES is implemented in C, and the source code is freely downloadable from <https://mrob.com/ries>. Your computer might already have a C or C++ compiler that you are unaware of, and there are several good free ones for Mac OS X, the Unix family, and Windows.

RIES optionally exploits 19-digit IEEE binary80 arithmetic if your CPU and C compiler support it. The resulting 19 digits is a valuable increment over the typical 16-digit hardware. IEEE binary80 is supported by Intel x86 processors, but not by current or past AMD or ARM processors. The Intel C++ compiler and the free GNU C compiler support this arithmetic, and Microsoft Visual C can be made to do so at the inconvenience of inserting `_controlfp_s` function calls at appropriate places in the source code.

Here is an example by Bill Gosper (private communication):

*Mathematica* 12.1 cannot determine a closed form for

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cos\left(\sqrt{\frac{1}{2} + \left(\frac{1}{2} + n\right)^2} \pi\right)}{\left(\frac{1}{2} + n\right) \left(\frac{1}{2} - \frac{1}{\sqrt{2}} + n\right) \left(\frac{1}{2} + \frac{1}{\sqrt{2}} + n\right)}$$

Suppose that you compute a 20-digit approximate value 7.0895773641597344051, and you invoke RIES as the command line

```
ries -l7 7.0895773641597344051 -s
```

where option `-l7` specifies the maximum search level and option `-s` permits implicit equational results that are not of the explicit form `x = constantExpression`. This feature is valuable because such equations are often solvable manually or by computer algebra, and you at least have a possibly useful different characterization of your wild constant that might provide insight and enable you to approximate it to larger precision more efficiently.

Here is the corresponding slightly edited RIES output:

Your target value: T = 7.08957736415973441 mrob.com/ries

```
x = pi + 4 for x = T + 0.0520153 {49}
x = 4 sqrt(pi) for x = T + 0.000238039 {58}
x = 1/(8 + pi) + 7 for x = T + 0.000176411 {81}
x = (sqrt(pi) + pi)/ln(2) for x = T - 0.000106841 {87}
x = e^(e^sqrt(phi))/5 for x = T - 8.60481e-05 {90}
x = ln(ln(6^8))^2 for x = T + 7.23949e-05 {93}
x = (phi^2)^sqrt(1 + pi) for x = T - 4.72474e-05 {85}
x = (e^(pi - 2))^2 - e for x = T + 1.25979e-05 {90}
x = (1/pi + sqrt(pi))^2 + e for x = T - 2.53137e-06 {92}
x = e (sinpi(1/ln(9)) + phi) for x = T + 2.0796e-06 {109}
x = 3/pi + 8 - pi^|x for x = T + 5.08817e-10 {113}
```

$\sin(\pi \sqrt{x}/e) = (1/(8/\phi - 1))^2$  for  $x = T - 3.68034e-10$  {131}  
 $x = 3((\pi^n/4 - 1/7) + \phi) - 2$  for  $x = T - 3.51782e-11$  {141}  
 $x = 2 e^n/(1 + e) - 1/(1 - 3^n/2)$  for  $x = T + 1.65079e-11$  {147}  
 $x = ((\log_9(1/2 + 8))/4)/e + 7$  for  $x = T - 1.3897e-11$  {150}  
 $x = -(\pi/\cos(\pi(1/\sqrt{2}))) - \pi \cos(\pi(1/\sqrt{2}))$  for  $x = T - 5.28657e-16$  {148}  
 $x = \pi (-1/\cos(\pi(1/\sqrt{2})) - \cos(\pi(1/\sqrt{2})))$  for  $x = T + 3.05745e-16$  {154}  
 $x = (1/\tan(\pi(1/\sqrt{8})))^4 + 1)(x/2 + \pi)$  for  $x = T - 1.30104e-18$  {164}

(Stopping now because best match is within 3.07e-18 of target value.)

$\log_A(B) = \text{logarithm to base A of B} = \ln(B) / \ln(A)$ ,

$\cos(\pi X) = \cos(\pi * x)$

$\sin(\pi X) = \sin(\pi * x)$

$\tan(\pi X) = \tan(\pi * x)$

$\phi = \text{the golden ratio} = (1 + \sqrt{5})/2$

$A^n/B = \text{Ath root of B}$

	--LHS--	--RHS--	-Total-	
max complexity:	86	81	167	
dead-ends:	363904522	872434773	1236339295	Time: 204.127 sec
expressions:	25836915	59427019	85263934	Memory: 1351040 KB
distinct:	7974910	9198455	17173365	
Total tested:	(7.336e+13)			

The integers in braces at the right ends of result lines are the complexities, which are computed differently from MESSearch. Notice that

- RIES rounded the 20 digit input to 1 digit less than the 19 digit arithmetic it is using on this computer.
- RIES displays a candidate only if its absolute difference from the entered float is less than that for the previously displayed candidate. Since candidates are generated in approximate order of non-decreasing complexity, the sequence of displayed candidates is approximately the Pareto set of optima for the conflicting objectives of large agreement and small complexity. Thus usually all but perhaps the last few of the displayed candidates are possibly-useful approximations rather than plausible candidates for the limit you seek. (But the last one or so can be incorrect because of overfits or the limited precision of the hardware arithmetic.)
- The last candidate is an implicit equation with  $x$  on both sides, but it can be solved exactly for  $x$  either manually or by computer algebra to give

$$x = -2\pi \left( 1 + \frac{2}{-1 + \cot\left(\frac{\pi}{2\sqrt{2}}\right)^4} \right) \simeq 7.0895773641597344057, \quad (9)$$

which is within a few 19-digit machine-epsilons of the input float.

- The nonequivalent candidate result implicit result  $\sin(\pi(\sqrt{x}/e)) = (1/(8/\phi - 1))^2$  is also solvable. RIES was able to explicitly invert the backward transformation for all of the other result lines.
- Both of the simpler explicit solutions immediately preceding the last output equation are equivalent to each other and to the explicit solution (9).<sup>8</sup>

RIES uses a particularly compact data structure that allows it to build particularly large tables before having to resort to slower secondary storage. Compiling with extended precision is slower and reduces the maximum achievable table sizes, but the greatly increased number of recognizable constants makes it worthwhile. To maximize your coverage and impostor detection, compile both a 19-digit and 16-digit version if your CPU supports that, then try both.

## 7 Functions built into computer algebra systems

### 7.1 The Maple `identify(...)` function

Sometimes you want to write software that invokes other software and uses the result without human intervention for that part. For that purpose, functions are usually better than downloadable or web applications because function invocation from software shouldn't require human keystrokes. Moreover, mathematical functions have a single output and should not have side effects, making them composable, with results that are the same regardless of temporal reorderings due to exploiting commutativity, associativity, etc. Functions are also more suitable for processing a batch of inputs without human intervention.

The Maple computer algebra system has a function named `identify(...)` that can take a float as an argument and returns either one nonfloat constant or the input float if the function cannot determine a nonfloat that the function judges sufficiently likely to be the limit you seek. This function is an adaptation by Kevin Hare of a Maple application by Alan Meichsner [12, 4], based on the PSLQ integer relation algorithm.

Default arbitrary-precision Maple floats have only 10-digit significands. Therefore you must almost always use more precision to have a reasonable chance of success.

The fact that `identify(...)` results are either the input float or a nonfloat that has very nearly the same value enables `identify(...)` to map automatically over the parts of non-real constants and over non-constant expressions, trying to replace all or at least some of any floats therein with close nonfloat constants. For example,

$$\text{identify} \left( \frac{0.27675082 - 1.0369278 I}{8.3118730x^2 + 11.94535128y} \right) \rightarrow \frac{0.27675082 - I\zeta(5)}{\sqrt{7}\pi x^2 + 4e \ln(3)y}$$

where  $I = \sqrt{-1}$  and  $\zeta$  is the Riemann zeta function.

---

<sup>8</sup>The simpler explicit equivalent results give the same 20 digit value despite their *estimated* absolute errors being about 100 times as large. This is probably because the estimated errors rely on linearized error analysis and/or the limitation to hardware floating-point prevents the float values of the nonfloat candidates from being computed more precisely than the float input.

The one float that identify did not convert was a random float. In contrast to applications that allow you to inspect alternatives and reject them all, a function such as `identify(...)` automatically replaces any number of floats in an expression by nonfloats without your inspection and consent to their complexity and agreement. It is therefore appropriate that the `identify` function is parsimonious and therefore did not replace the random float with a candidate that would almost certainly be an impostor.

To determine your confidence in a Maple `identify` result, try using successively increasing precisions for the input floats until the result appears to stabilize. Then, assess the agreements of the replaced floats with their replacements by using the `evalf` function on each of those replacements.<sup>9</sup>

The `identify` function has optional arguments that enable you to adjust the integer relation models, and it is well worth doing so if the defaults do not return a satisfactory result. For example, although the default maximum degree for seeking algebraic numbers such as `RootOf [polynomial, ...]` is 6, you can change that. Many algebraic numbers of interest have larger degree, so try larger degrees if an algebraic number is plausible and you can compute your wild float to high precision.

## 7.2 The MPMath, SymPy and Sage `identify(...)`, `findpoly(...)`, and `nsimplify(...)`, functions

Sage is a software system freely downloadable from <http://www.sagemath.org/> that includes NumPy, SciPy, matplotlib, SymPy, Maxima, GAP, FLINT, R and other packages. The SymPy computer algebra system in turn includes MPMath, which has an `identify(...)` function analogous to Maple's, except that identification of algebraic numbers is done by a separate function named `findpoly(...)`. The SymPy system also has an `nsimplify(...)` function somewhat analogous to the `NSimplify [...]` component of `AskConstants` described in subsection 8.2.

# 8 Functions and applications for CAS

## 8.1 An exhaustive search toolkit for *Mathematica*

Andrzej Odrzywołek analyzes the probability of correctly identifying a float constant using forward exhaustive search [13]. In support of that research he implemented a toolkit in *Mathematica* freely downloadable from

<https://github.com/VA00/SymbolicRegressionPackage>

It includes a unidirectional `RecognizeConstant` function. Your initial components can include any *Mathematica* symbolic constant that can be evaluated to a float or any *Mathematica* function that returns a float result for float inputs. Being unidirectional, it is not as fast or memory efficient as `MESearch` and `RIES`.

---

<sup>9</sup>A replacement subexpression could still be an impostor, but if you are satisfied with the float values of the replacements, then at least the result is a satisfactory approximation to the original float, which was not an exact representation either.

My favorite function in the toolkit is a `RandomExpression` function that returns a random expression of a requested complexity composed of specified constants, variables, functions and operators. Being exhaustive, it is particularly appropriate for generating examples for which `MESearch` and `RIES` described in Subsection 6.2 are particularly successful.

Another useful function is `EnumerateExpressions`, which generates all expressions of a specified complexity composed of specified components.

## 8.2 AskConstants and associated functions for *Mathematica*

`AskConstants` is an application that I implemented in *Mathematica*, freely downloadable from [AskConstants.org](http://AskConstants.org). `AskConstants 5.0` has about 3000 integer relation models, and also bidirectional search with a choice of precomputed tables for which the largest backward table has about 5 million entries and the largest forward table has about 15 million entries. The table lookup exploits sign and base 2 significand aliasing, with automatic de-aliasing and inversion for close matches.<sup>10</sup> Thus ignoring overflow underflow, nonreal compositions and the aliasing bonus, the largest tables cover about  $(5 \times 10^6)(15 \times 10^6) = 7.5 \times 10^{13}$  expressions. The backward tables and about half of the largest forward table use tabulated and published constants. The other half of the largest tables was generated by exhaustive table building with the elementary functions and the most commonly-occurring special functions. This hybrid combines advantages of precomputed tables such as `ISC` with those of exhaustive breadth-first tables such as `MESearch` and `RIES`.

Figure 4 shows the use of version 5.0 to propose a closed form for the definite integral

$$\int_0^{\infty} \frac{dx}{\sqrt{1+x^2} + (1+x^2)^{7/2}}$$

that the *Mathematica* 12.1 `Integrate` function cannot determine.<sup>11</sup>

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<sup>10</sup>The bonus for base 2 aliasing exceeds that of base 10 aliasing because 2 divides a random reduced numerator or denominator more often than 10 does, and forward table entries of the form *rational*  $\times$  *irrational* only need to include positive *rational* numbers having an odd numerator and denominator: The appropriate sign and power of 2 are recovered by the automatic dealiasing. Also, without loss of generality, backward transformations of the form *rational*  $\times$  *anyForwardTableEntry* can limit the *rational* to being positive with an odd numerator and denominator.

<sup>11</sup>I thank Daniel Lichtblau for this example.



1. The float value of the input numerical integration is echoed under the digit ruler, with the darker digits estimated to be correctly rounded by significance arithmetic and with the normally undisplayed two agreeing guard digits displayed gray.
2. The upper horizontal dashed plot line is the Precision of the input float as estimated by significance arithmetic.
3. The **Agreement** of a candidate nonfloat is **the number of digits that agree** with the input float within 1/2 unit, up to the length of the input significand, including any stored digits that are not ordinarily displayed.
4. The lower horizontal dashed line is the smallest allowable Agreement for displaying a nonfloat on the left and plotting it as a large dot on the scatter plot.
5. **Entropy10** based on information theory is a measure of nonfloat expression complexity that is the sum of the base-ten logarithms of the absolute values of all the nonzero integers occurring in numerators and denominators in an expression, plus about 1.0 per occurrence of a named constant, function or operator – counting duplicates. The implicit units of Entropy10 are also digits.
6. It is desirable to have high agreement, but it is also desirable to have low expression complexity because floats can be approximated arbitrarily well by sufficiently complicated nonfloats that are impostors rather than the limit you seek.
7. Therefore

$$\text{Margin} := \text{Agreement} - \text{Entropy10} \tag{11}$$

combines these two competing objectives into one of maximizing Margin, with the largest Margin in the upper left corner of the plot and the smallest Margin in the lower right corner.

8. The dashed diagonal line is the smallest allowable Margin for displaying a candidate nonfloat on the left side of the output.
9. Plot points in the upper leftmost pink region are very rarely impostors, Plot points in the lower right white region are almost always impostors, with other background colors having likelihoods that vary monotonically between these extremes.
10. As illustrated, hovering the mouse over a plot point invokes a tooltip listing the nonfloat candidate together with its Entropy10, Agreement and Margin. The latter two include a qualitative likelihood estimate ranging adjectives “Excellent” through “Terrible” based on my experience with test examples.<sup>12</sup>

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<sup>12</sup>Candidates are not produced in approximate order of increasing complexity, so the helpful numeric likelihood estimates of MESSearch are inapplicable.



11. Hovering over a small plot point of a rejected candidate sometimes reveals a plausible candidate for which the precision of the input was insufficient to meet the minimum acceptable agreement and margin. If it is the limit you seek, then increasing input precision enough should move it vertically up into the accepted regions.
12. More generally there might be several accepted candidates. Ones judged equivalent by applying the *Mathematica* PossibleZeroQ function to their differences are joined by line segments, which can help you assess them more efficiently.

Automatic use of the backward transformation requires an inverse function for the top-level function in the transformation; and as with most mathematics software, *Mathematica* has very few inverse special functions – probably because they are extremely difficult to implement for nonreal arguments. However, like all of the other tools discussed in this article, AskConstants directly addresses only real floats, and it is not too difficult to implement most inverse special functions for real arguments and results. Consequently I did that for about 40 special functions. Many of those are multi-branched, such as for BesselJ, Gamma, and Zeta. Consequently I also implemented corresponding functions that return the abscissa and ordinates of the local infima and suprema of those functions to accomplish piecewise monotonic partitions of the functions being inverted. The real inverse and infima or suprema functions are separately usable, as are some additional functions analogous to the *Mathematica* BesselJZero function.<sup>13</sup>

To overcome the limitation to constant expressions, real arguments and results, AskConstants also contains a ProposeBestOrInput function that maps over general expressions and over the real and imaginary parts of non-real constants similar to the Maple identify function.

### 8.2.1 NSimplify [...]

Anyone who tests the programs described here with constants having known nonfloat representations might eventually notice that surprisingly often an equivalent proposed candidate is simpler than the published expression. This is because:

- Manual derivations and default computer algebra simplification often produce incompletely simplified expressions.
- Sometimes there is no composition of builtin optional computer algebra transformations that can produce a particularly simple representable equivalent that exists.

Consequently the AskConstants download includes an NSimplify function that supplements the *Mathematica* FullSimplify function by approximating a nonfloat input expression as a float, then applying the ProposeBestOrInput function to that float. If it returns a nonfloat having smaller Entropy10 than the given nonfloat and the *Mathematica* PossibleZeroQ function judges that the difference between the original expression and proposed replacement is 0, then the original is replaced. If that doesn't succeed, then

---

<sup>13</sup>Some users might find these functions more frequently useful than AskConstants.

this process is recursively applied to subexpressions. NSimplify also tries FullSimplify so that the result is at least that successful. This brute-force combination is slow, but it can achieve some dramatic simplifications. Some users might find NSimplify more frequently useful than AskConstants and ProposeBestOrInput.

As an example, applying NSimplify to

$$\sqrt{\frac{1}{2} - \frac{1}{4 \sqrt{4 + \sqrt{7 - \sqrt{5}} + \sqrt{30 - 6\sqrt{5}}}}}} \sqrt[4]{\frac{1}{2}}$$

Root [-97 + 448 #1 - 672 #1<sup>2</sup> + 560 #1<sup>3</sup> - 280 #1<sup>4</sup> + 84 #1<sup>5</sup> - 14 #1<sup>6</sup> + #1<sup>7</sup> &, 1]

produced the result

$$\frac{\sin\left(\frac{7\pi}{120}\right)}{2 - 31^{1/7}};$$

and the optional trace reported the steps

- Level 2: ProposeBestOrInput  $\left[ N \left[ \frac{1}{2} - \sqrt{\frac{1}{4\sqrt{\dots}}}, 26 \right] \right] \mapsto \sin\left(\frac{7\pi}{120}\right)$ ,  
reducing Entropy10 by 14.1; and PossibleZeroQ [difference]  $\mapsto$  True.
- Level 2: ProposeBestOrInput  $[N [\text{Root} [-97 + \dots + \#1^7 \&, 1]], 38] \mapsto \frac{1}{2 - 31^{1/7}}$ ,  
reducing Entropy10 by 18.6; and PossibleZeroQ [difference]  $\mapsto$  True.

*Remarks:*

1. NSimplify maps over over general expressions, replacing exact constant subexpressions with less complex ones where it can.
2. The *Mathematica* PossibleZeroQ Function can incorrectly return False, but this would merely cause NSimplify to miss an Entropy10 reduction opportunity.
3. The PossibleZeroQ function can incorrectly return True, causing NSimplify to return a nonequivalent result. However, in this application that also requires the parsimonious AskConstants ProposeBestOrInput function to have generated an impostor nonfloat for a float subexpression, and the product of these two small probabilities is such that I have not yet noticed it happen. I suspect that the overall probability is not much more than that of a nonequivalent result due to a bug in the many built-in *Mathematica* functions.

### 8.3 Plouffe's inverter for Maple

After developing ISC, Simon Plouffe developed Plouffe's Inverter [15], a similar Maple application with larger tables. He has several versions using different significand sizes, and over the years his tables have grown. As of January 2022 his largest forward table has about  $8 \times 10^{10}$  entries.

The same site <https://archive.org/> (not arXiv.org) that hosts ISC has agreed to also store at least one of his versions for downloading. For downloading, all three versions require you to have Maple, a fast reliable download connection and a computer with a large amount of secondary storage.

Therefore, I am hoping that the site will also host a free web application alongside the ISC that they already host.

## 9 Custom Integer Relation Models

Wild floats often occur at the high end of a family of problems depending on a parameter  $n$ , with known nonfloat values for small  $n$ . For example, a noticeable number of definite integrals have nonfloat representations that can be expressed as a rational linear combination of terms with cofactors that are small positive integer powers of  $\pi$ ,  $\ln 2$ ,  $\ln 3$ ,  $\zeta(m)$  with small integer  $m > 1$ , and low-order polylogarithms having simple arguments such as  $1/2$  or  $1/4$ , perhaps multiplied by  $\sqrt{2}$  or  $\sqrt{3}$ . The known closed forms for small  $n$  permits guesses of possible cofactors for the next value of  $n$ , and if a guess is successful, then you can use that form to make cofactor guesses for the next value of  $n$ , and so on.

An increasing number of computer algebra systems have builtin integer-relation solvers. David Bailey and David Broadhurst [1, 2] describe some particularly efficient implementations in Fortran 90 and C++. Such large models are too time consuming to make them part of the set of more general-purpose integer relation models in tools such as WolframAlpha, AskConstants, or the Maple and MPMath identify functions. However, it is easy to simply invoke a built-in or stand-alone PSLQ function with an input vector containing your float and a set of cofactors. The LLL algorithm can also be used for this purpose if that is available instead.

## 10 The curse of extreme magnitude

The software described here is most successful at proposing candidate nonfloats for floats whose magnitudes are not extremely different from 1.0. A reason for this is that representations of nonfloat constants having extreme magnitudes often require high-complexity extreme magnitudes of the numerators or denominators of some rational numbers therein, which requires extreme precision input floats for integer relation algorithms or prohibitively extreme magnitude integer subexpressions in lookup tables. Exceptions are functions whose nonzero magnitude can be extremely large or small for arguments that are not extremely large or small, such as  $\Gamma(98/3) \simeq 8.26 \times 10^{34}$  or  $\operatorname{erfc}(9) \simeq 4.14 \times 10^{-37}$ .

Fortunately, most published *mathematics* constants do not have extreme magnitudes. For example, in Steven Finch's table of about 10,000 such constants, nonzero magnitudes vary from 0.000111582 through 137.0359 with median about 0.9 and quartiles about 0.4 and 1.9.

## 11 Some causes of impostors

“1.0000001 is a crowd”  
– adapted from James Thurber

Reasons for impostors include:

1. Many functions  $f(x)$  have a stationary value of 1 for some value of  $x$ . For example,  $\cos x$ ,  $\sec x$  and  $\cosh x$ , at  $x = 0$ ; or  $\tanh x$  and  $\operatorname{erf}(x)$  as  $x \rightarrow \infty$ . In a neighborhood of the stationary point, such constants that are modeled are likely to occur as impostors for any that are not modeled, because very low complexity  $x$  can produce values very close to 1.0 and hence each other. For example,  $\sec(1/999)$  and  $\cosh(1/999)$  differ by only 2 units in the 14th place. Therefore, if only one of these was modeled, you could easily get one of them as an impostor for the other.
2. If a sum has some terms with very small magnitudes compared to other terms, impostors often omit those small-magnitude terms – at least until sufficiently large precision is used.
3. If for some relatively low complexity nonfloat constant  $x$  a modeled expression  $f(x)$  agrees relatively closely with an unmodeled nonequivalent expression  $g(x)$ , then that makes it easy for the modeled expression to be an impostor for the unmodeled expression. For example the low complexity expressions,  $e^{18}$ ,  $2 \sinh 18$ , and  $2 \cosh 18$  differ by only 1 unit in the last of 16 places. If a proper subset of these is modeled and the true limit is an unmodeled one of these, then one of the modeled ones will almost certainly occur as an impostor having nearly the same agreement and complexity as the correct limit.

## 12 Conclusions

There are many good tools that can propose nonfloat candidates that your float closely approximates. Usefully often one of those candidates is the limit that your float would approach as the working precision increases. Some of the tools are easy to use directly on the internet, some are built-into a computer algebra system, and others are easy to download and install.

Most of the programs discussed in this article can propose correct candidates that none of the others can propose. General-purpose web search engines, radix search, integer relation models, precomputed tables and exhaustive run-time tables are each best at overlapping kinds of expressions. Therefore it is worthwhile to try as many of these programs as is reasonably convenient.

Consequently it is wise to try at least:

- the Online Encyclopedia of Integer Sequences<sup>14</sup>,

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<sup>14</sup>

– This is also a good place to find high precision values and program fragments for efficiently computing more digits.

- at least one of the general purpose web search engines on your computers or smart phones,
- one of the tools that have numerous integer-relation models and can identify algebraic numbers, preferably arbitrary degree, returning results such as  $\text{Root}[\text{polynomial}, n]$  : AskConstants, Maple identify, SymPy identify and FindPoly, or WolframAlpha,
- one that has precomputed lookup tables (AskConstants, Inverse Symbolic Calculator, and Plouffe’s Inverter),
- one that uses exhaustive bidirectional breadth-first search (MESearch and RIES).

At the very least you should try all that are built into or downloadable for the computer algebra systems that you already have, together with all of the web-based tools (a browser’s search engine, the Online Encyclopedia of Integer Sequences, Inverse Symbolic Calculator and WolframAlpha) because that is so easy to do.

You might develop favorites that tend to work better for your needs, but it is nice to have all of these alternatives when your favorites don’t succeed and the result is important to you.

The success of your efforts depends strongly on knowing how to groom your float for input, interpret the results, and possibly de-alias and transform the result. These details vary greatly among the tools, as described in this article.

When none of these programs accessible to you can propose a plausible constant, then at least you can have the peace of mind of greatly reducing the chance that your problem has a simple nonfloat result that you did not find.

## Acknowledgments

Thank you Bill Gosper, Daniel Lichtblau, Robert Munafo, Simon Plouffe, Neil Sloane and Michael Trott for your helpful suggestions.

## References

- [1] Bailey, D. H., The PSLQ Algorithm: Techniques for Efficient Computation (slides), 2010, <https://www.davidhbailey.com/dhbtalks/dhb-carma-20100824.pdf>
- [2] Bailey, D. H. and Broadhurst, D. J., Parallel Integer Relation Detection: Techniques and Applications, 1999, <https://arxiv.org/pdf/math/9905048.pdf>
- [3] Borwein, J. and Borwein, P., *A Dictionary of Real Numbers*, Wadsworth Inc., 1990.
- [4] Borwein, P., Hare, K. G., and Meichsner, A., Reverse Symbolic computations, the identify function, [https://www.researchgate.net/publication/267425704\\_REVERSE\\_SYMBOLIC\\_COMPUTATIONS\\_THE\\_IDENTIFY\\_FUNCTION](https://www.researchgate.net/publication/267425704_REVERSE_SYMBOLIC_COMPUTATIONS_THE_IDENTIFY_FUNCTION)

- [5] Corless, R, Gonnet, G., Hare, D., Jeffrey, D., Knuth, On the Lambert W function, *Advances in Computational Mathematics*, 5, (1996), pp. 329–359.
- [6] Dougherty-Bliss, R. and Zeilberger, D., Automatic conjecturing and proving of exact values of some infinite families of infinite continued fractions, 2020, <https://arxiv.org/pdf/2004.00090.pdf>
- [7] Ferguson, H. R. P. and Bailey, D. H., A Polynomial Time, Numerically Stable Integer Relation Algorithm, RNR Technical Report RNR-91-032, July 14, 1992.
- [8] Ferguson, H. R. P., Bailey, D. H., and Arno, S., Analysis of PSLQ, an integer relation finding algorithm, *Mathematics of Computation* 68 (1999) 351–369.
- [9] Finch, S. R., *Mathematical Constants, Encyclopedia of Mathematics and its Applications*, Cambridge University Press, 2003.
- [10] Finch, S. R., *Mathematical Constants II, Encyclopedia of Mathematics and its Applications*, Cambridge University Press, 2018.
- [11] Johansson, F., Ordner: index of real numbers, <https://fungrim.org/ordner>
- [12] Meichsner, A., *Integer relation algorithms and the recognition of numerical constants*, M.S. Thesis, Simon Fraser University, 2001. <http://www.collectionscanada.gc.ca/obj/s4/f2/dsk3/ftp04/MQ61592.pdf>
- [13] Odrzywolek, A., Criteria for the numerical constant recognition, 2020, <https://arxiv.org/abs/2002.12690>
- [14] Plouffe, S., Credits and References for Inverse Symbolic Calculator. <http://wayback.cecm.sfu.ca/projects/ISC/credits.html>, 1995.
- [15] Plouffe, S., L’Inverseur, March 1998, <http://vixra.org/pdf/1409.0151v1.pdf>
- [16] Raayoni, G. et al., The Ramanujan Machine: Automatically generated conjectures on fundamental constants, 2020, <https://arxiv.org/pdf/1907.00205.pdf>
- [17] Robinson, H. P. and Potter, E., *Mathematical Constants*, UCRL-20418, UC-32 Math. and Comp., TID-4500 (57th Ed.), March 1971.
- [18] Salsamendi, J. Z., An efficient mathematical expression searcher for constant recognition and some conjectures and theorems discovered with it, 2013, <https://web.archive.org/web/20180728144309/http://xuru.org/downloads/papers/MESearch.pdf>
- [19] Salvy, B. and Zimmermann, P., GFUN: a Maple package for the manipulation of generating and holonomic functions in one variable, *ACM Transactions on Mathematical Software*, 20(2), June 1994, pp. 163-177.
- [20] Shamos, M. I., Shamos’s Catalog of the Real Numbers, <https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.366.9997&rep=rep1&type=pdf>

- [21] Stoutemyer, D. R., The Constant Hunters, *Applications of Computer Algebra Conference*, 2019, Montreal, <http://aca2019.etsmtl.ca/program/invited-speakers/>.